

SIMPLE MIXING ACTIONS WITH UNCOUNTABLY MANY PRIME FACTORS

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ABSTRACT. Via (C, F) -construction we produce a 2-fold simple mixing transformation which has uncountably many non-trivial proper factors and all of them are prime.

0. INTRODUCTION

This paper is about prime factors of simple probability preserving actions. We first recall these and related definitions from the theory of joinings.

Let $T = (T_g)_{g \in \Gamma}$ be an ergodic action of a locally compact second countable group Γ on a standard probability space (X, \mathfrak{B}, μ) . The main interest for us lies in \mathbb{Z} and \mathbb{R} -actions. A measure λ on $X \times X$ is called a *2-fold self-joining* of T if it is $(T_g \times T_g)_{g \in \Gamma}$ -invariant and it projects onto μ on both coordinates. Denote by $J_2^e(T)$ the set of all ergodic 2-fold self-joinings of T . Let $C(T)$ stand for the centralizer of T , i.e. the set of all μ -preserving invertible transformations of X commuting with T_g for each $g \in \Gamma$. Given a transformation $S \in C(T)$, we denote by μ_S the corresponding *off-diagonal measure* on $X \times X$ defined by $\mu_S(A \times B) := \mu(A \cap S^{-1}B)$ for all $A, B \in \mathfrak{B}$. In other words, μ_S is the image of the measure μ under the map $x \mapsto (x, Sx)$. Of course, $\mu_S \in J_2^e(T)$ for every $S \in C(T)$. If T is weakly mixing, $\mu \times \mu$ is also an ergodic self-joining. If $J_2^e(T) \subset \{\mu_S \mid S \in C(T)\} \cup \{\mu \times \mu\}$ then T is called *2-fold simple* [Ve], [dJR]. By a *factor* of T we mean a non-trivial proper T -invariant sub- σ -algebra of \mathfrak{B} . If T has no non-trivial proper factors then T is called *prime*. In [Ve] it was shown that if T is 2-fold simple then for each non-trivial factor \mathfrak{F} of T there exists a compact (in the strong operator topology) subgroup $K_{\mathfrak{F}} \subset C(T)$ such that $\mathfrak{F} = \mathfrak{F}_{K_{\mathfrak{F}}}$, where

$$\mathfrak{F}_K = \{A \in \mathfrak{B} \mid \mu(kA \triangle A) = 0 \text{ for all } k \in K\}$$

is the fixed algebra of K . In particular, \mathfrak{F} (or, more precisely, the restriction of T to \mathfrak{F}) is prime if and only if $K_{\mathfrak{F}}$ is a maximal compact subgroup of $C(T)$.

One of the natural questions arising after the general theory of simple actions was developed in [dJR] is: are there simple maps with non-unique prime factors? The first example of such maps was constructed by Glasner and Weiss [GIW] as an inverse limit of certain horocycle flows. For that they used some subtle facts from Ratner's theory of joinings for horocycle flows and properties of lattices in $SL_2(\mathbb{R})$. The authors of a later paper [DdJ] utilized a more elementary cutting-and-stacking technique to construct a weakly mixing 2-fold simple transformation which has countably many factors, all of which are prime. Our purpose in the present paper is to use a similar cutting-and-stacking technique to produce a *mixing* transformation which has *uncountably* many factors, all of which are prime.

Via (C, F) -construction we produce a measure preserving action T of an auxiliary group $G = \mathbb{Z} \times (\mathbb{R} \rtimes \mathbb{Z}_2)$ such that the transformation $T_{(1,0,0)}$ is mixing 2-fold simple and $C(T_{(1,0,0)}) = \{T_g \mid g \in G\}$. Since all non-trivial compact subgroups of G are $G_b = \{(0, 0, 0), (0, b, 1)\}$, $0 \neq b \in \mathbb{R}$, and all of them are maximal, this gives an example of 2-fold simple transformation with uncountably many prime factors. All these factors are 2-to-1 and pairwise isomorphic.

We also correct a gap in the proof of [DdJ, Lemma 2.3(ii)] (see Remark 2.4).

The skeleton of the proof of the main result is basically the same as in [DdJ], where the “discrete case” (i.e. the auxiliary group is discrete) was under consideration. To work with the (C, F) -construction for actions of continuous (i.e. non-discrete) groups we use the approximation techniques from [Da2].

1. (C, F) -CONSTRUCTION

We now briefly outline the (C, F) -construction of measure preserving actions for locally compact groups. For details see [Da1] and references therein.

Let G be a unimodular locally compact second countable (l.c.s.c.) amenable group. Fix a $(\sigma$ -finite) Haar measure λ on it. Given two subsets $E, F \subset G$, by EF we mean their algebraic product, i.e. $EF = \{ef \mid e \in E, f \in F\}$. The set $\{e^{-1} \mid e \in E\}$ is denoted by E^{-1} . If E is a singleton, say $E = \{e\}$, then we will write eF for EF . Given a finite set A , $|A|$ will denote the cardinality of A . Given a subset $F \subset G$ of finite Haar measure, λ_F will denote the probability on F given by $\lambda_F(A) := \lambda(A)/\lambda(F)$ for each measurable $A \subset F$. If D is finite, then κ_D will denote the equidistributed probability on D , that is $\kappa_D(A) := |A|/|D|$ for each $A \subset D$.

To define a (C, F) -action of G we need two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ of subsets in G such that the following are satisfied:

$$(1.1) \quad (F_n)_{n=0}^\infty \text{ is a Følner sequence in } G,$$

$$(1.2) \quad C_n \text{ is finite and } |C_n| > 1,$$

$$(1.3) \quad F_n C_{n+1} \subset F_{n+1},$$

$$(1.4) \quad F_n c \cap F_n c' = \emptyset \text{ for all } c \neq c' \in C_{n+1}.$$

This means that $F_n C_{n+1}$ consists of $|C_{n+1}|$ mutually disjoint ‘copies’ $F_n c$, $c \in C_{n+1}$, of F_n and all these copies are contained in F_{n+1} .

First, we define a probability space (X, μ) in the following way. We equip F_n with the measure $(|C_1| \cdots |C_n|)^{-1} \lambda \upharpoonright F_n$ and endow C_n with the equidistributed probability measure. Let $X_n := F_n \times \prod_{k>n} C_k$ stand for the product of measure spaces. Define an embedding $X_n \rightarrow X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n c_{n+1}, c_{n+2}, \dots).$$

It is easy to see that this embedding is measure preserving. Then $X_1 \subset X_2 \subset \dots$. Let $X := \bigcup_{n=0}^\infty X_n$ denote the inductive limit of the sequence of measure spaces X_n and let \mathfrak{B} and μ denote the corresponding Borel σ -algebra and measure on X respectively. Then X is a standard Borel space and μ is σ -finite. It is easy to check that μ is finite if and only if

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{\lambda(F_n)}{|C_1| \cdots |C_n|} < \infty.$$

If (1.5) is satisfied then we choose (i.e., normalize) λ in such a way that $\mu(X) = 1$.

Now we define a μ -preserving action of G on X . Suppose that the following is satisfied:

$$(1.6) \quad \text{for any } g \in G, \text{ there is } m \geq 0 \text{ with } gF_n C_{n+1} \subset F_{n+1} \text{ for all } n \geq m.$$

For such n , take $x \in X_n \subset X$ and write the expansion $x = (f_n, c_{n+1}, c_{n+2}, \dots)$ with $f_n \in F_n$ and $c_i \in C_i$, $i > n$. Then we let

$$T_g x := (g f_n c_{n+1}, c_{n+2}, \dots) \in X_{n+1} \subset X.$$

It follows from (1.6) that T_g is a well defined μ -preserving transformation of X . Moreover, $T_g T_h = T_{gh}$, i.e. $T := (T_g)_{g \in G}$ is a μ -preserving Borel action of G on X . T is called the (C, F) -action of G associated with $(C_{n+1}, F_n)_{n=0}^\infty$.

We now recall some basic properties of $(X, \mathfrak{B}, \mu, T)$. Given a Borel subset $A \subset F_n$, we put

$$[A]_n := \{x \in X \mid x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \text{ and } f_n \in A\}$$

and call this set an n -cylinder. It is clear that the σ -algebra \mathfrak{B} is generated by the family of all cylinders. Given Borel subsets $A, B \subset F_n$, we have

$$(1.7) \quad [A \cap B]_n = [A]_n \cap [B]_n, [A \cup B]_n = [A]_n \cup [B]_n,$$

$$(1.8) \quad [A]_n = [AC_{n+1}]_{n+1} = \bigsqcup_{c \in C_{n+1}} [Ac]_{n+1},$$

$$(1.9) \quad \mu([A]_n) = |C_{n+1}| \mu([Ac]_{n+1}) \text{ for every } c \in C_{n+1},$$

$$(1.10) \quad \mu([A]_n) = \mu(X_n) \lambda_{F_n}(A),$$

$$(1.11) \quad T_g[A]_n = [gA]_n \text{ if } gA \subset F_n,$$

$$(1.12) \quad T_g[A]_n = T_h[h^{-1}gA]_n \text{ if } h^{-1}gA \subset F_n.$$

Each (C, F) -action is of funny rank one (for the definition see [Fe] for the case of \mathbb{Z} -actions and [So] for the general case) and hence ergodic. It also follows from (1.2) that T is conservative.

2. MAIN RESULT

By \mathbb{Z}_n we denote a cyclic group of order n , i.e. $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. Let $G := \mathbb{Z} \times (\mathbb{R} \rtimes \mathbb{Z}_2)$ with multiplication law as follows

$$(x, a, n)(y, b, m) := (x + y, a + (-1)^n b, n + m).$$

Then the center $C(G)$ of G is $\mathbb{Z} \times \{0\} \times \{0\}$. Each compact subgroup of G coincides with $G_b = \{(0, 0, 0), (0, b, 1)\}$ for some $b \in \mathbb{R}$. Notice that G_b is a maximal compact subgroup of G if $b \neq 0$.

To construct the required (C, F) -action of G we will determine a sequence $(C_{n+1}, F_n)_{n=0}^\infty$. Let $(r_n)_{n=0}^\infty$ be an increasing sequence of positive integers such that

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{n^4}{r_n} = 0.$$

Below — just after Lemma 2.1 — one more restriction on the growth of $(r_n)_{n=0}^\infty$ will be imposed and we will assume that r_n is large so that (2.6) is satisfied. We define recurrently three other sequences $(\tilde{a}_n)_{n=0}^\infty$, $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of positive integers by setting

$$\tilde{a}_0 := 1,$$

$$\begin{aligned}
a_n &:= (2r_{n-1} + 1)\tilde{a}_{n-1} \text{ for } n \geq 1, \\
b_n &:= (2n - 1)\tilde{a}_{n-1} \text{ for } n \geq 1, \\
\tilde{a}_n &:= a_n + b_n + n \text{ for } n \geq 1.
\end{aligned}$$

For each $n \in \mathbb{N}$, we let

$$\begin{aligned}
I_n &:= \{-n, \dots, n\}^2 \subset \mathbb{Z}^2, \\
H_n &:= \{-r_n, \dots, r_n\}^2 \subset \mathbb{Z}^2, \\
F_n &:= (-a_n, a_n]_{\mathbb{Z}} \times (-a_n, a_n]_{\mathbb{R}} \times \mathbb{Z}_2, \\
S_n &:= (-b_n, b_n]_{\mathbb{Z}} \times (-b_n, b_n]_{\mathbb{R}} \times \mathbb{Z}_2, \\
\tilde{F}_n &:= (-\tilde{a}_n, \tilde{a}_n]_{\mathbb{Z}} \times (-\tilde{a}_n, \tilde{a}_n]_{\mathbb{R}} \times \mathbb{Z}_2.
\end{aligned}$$

We also consider a homomorphism $\phi_n: \mathbb{Z}^2 \rightarrow G$ given by

$$\phi_n(i, j) := (2i\tilde{a}_n, 2j\tilde{a}_n, 0).$$

We then have

$$(2.2) \quad S_n \subset F_n, \quad F_n S_n = S_n F_n \subset \tilde{F}_n \subset G,$$

$$(2.3) \quad S_{n+1} = \tilde{F}_n \phi_n(I_n) = \bigsqcup_{h \in I_n} \tilde{F}_n \phi_n(h) = \bigsqcup_{h \in I_n} \phi_n(h) \tilde{F}_n,$$

$$(2.4) \quad F_{n+1} = \tilde{F}_n \phi_n(H_n) = \bigsqcup_{h \in H_n} \tilde{F}_n \phi_n(h) = \bigsqcup_{h \in H_n} \phi_n(h) \tilde{F}_n,$$

Suppose also that F_n is equipped with a finite partition ξ_n such that the following are satisfied:

- (i) the diameter of each atom of ξ_n is less than $\frac{1}{n}$,
- (ii) for each atom $A \in \xi_{n-1}$ and each element $c \in C_n$, the subset $Ac \subset F_n$ is ξ_n -measurable and
- (iii) ξ_n is symmetric, that is $A^{-1} \in \xi_n$ whenever $A \in \xi_n$.

It follows that for each measurable subset $A \subset F_n$, any $\varepsilon > 0$ and for all k large enough, there is a ξ_k -measurable subset $B \subset F_k$ such that $\mu([A]_n \triangle [B]_k) < \varepsilon$. We will denote by $\sigma(\xi_n)$ the σ -algebra on F_n generated by ξ_n .

For a finite subset D in S_n , we denote by κ_D the corresponding normalized Dirac comb, i.e. a measure on S_n given by $\kappa_D(A) := \frac{|A \cap D|}{|D|}$ for each subset $A \subset S_n$. Given two subsets $A, B \subset F_n$ define a function $f_{A,B}: S_n \times S_n \rightarrow \mathbb{R}$ by setting $f_{A,B}(x, y) := \frac{\lambda(Ax \cap By)}{\lambda(F_n)}$, $x, y \in S_n$. Choose a finite subset D_n in S_n such that

$$(2.5) \quad \left| \int_{S_n \times S_n} f_{Ag, Bh} d\kappa_{D_n} d\kappa_{D_n} - \frac{1}{\lambda(S_n)^2} \int_{S_n \times S_n} f_{Ag, Bh} d\lambda d\lambda \right| < \frac{1}{n}$$

for each ξ_n -measurable subsets $A, B \subset F_n$ and any $g, h \in F_n$ with $AgS_n, BhS_n \subset F_n$. For instance, let ξ_n consists of ‘rectangles’ $\{a\} \times \Delta \times \{m\} \subset G$, where $a \in (-a_n, a_n]_{\mathbb{Z}}$, $m \in \mathbb{Z}_2$ and $\Delta \subset (-a_n, a_n]_{\mathbb{R}}$ is a subinterval of length n^{-1} , and set $D_n := \{(a, kn^{-2}, m) \mid a \in (-b_n, b_n]_{\mathbb{Z}}, k \in (-n^2 b_n, n^2 b_n]_{\mathbb{Z}}, m \in \mathbb{Z}_2\}$. It is an easy exercise to check that (2.5) is satisfied for such ξ_n and D_n . We notice also that $|D_n^0| = |D_n^1|$.

Given a finite (signed) measure ν on a finite set D , we let $\|\nu\|_1 := \sum_{d \in D} |\nu(d)|$. If $\pi: D \rightarrow E$ then clearly $\|\nu \circ \pi^{-1}\|_1 \leq \|\nu\|_1$. Given a finite set Y and a mapping $s: Y \rightarrow D$, let $\text{dist}_{y \in Y} s(y)$ denote the image of the equidistribution on Y under s :

$$\text{dist}_{y \in Y} s(y) := \frac{1}{|Y|} \sum_{y \in Y} \delta_{s(y)} = \kappa_D \circ s^{-1}.$$

The following lemma easily follows from [dJ, Lemma 2.1] (cf. [Da2, Lemma 3.2]).

Lemma 2.1. *Let D be a finite set. Then given $\varepsilon > 0$ and $\delta > 0$, there is $R \in \mathbb{N}$ such that for each $r > R$, there exists a map $s: \{-r, \dots, r\}^2 \rightarrow D$ such that*

$$\|\text{dist}_{0 \leq t < N}(s_n(h + (t, 0)), s_n(h' + (t, 0))) - \kappa_D \times \kappa_D\|_1 < \varepsilon$$

for each $N > \delta r$ and $h \neq h' \in \{-r, \dots, r\}^2$ with $h_1 + N < r$ and $h'_1 + N < r$.¹

Applying this Lemma with $\varepsilon = \frac{1}{n}$ and $\delta = \frac{1}{n^2}$ we get the following. If r_n is large enough then there is a mapping $s_n: H_n \rightarrow D_n$ such that for any $N > \frac{r_n}{n^2}$ and $h \neq h' \in H_n \cap (H_n - (N - 1, 0))$ we have

$$(2.6) \quad \|\text{dist}_{0 \leq t < N}(s_n(h + (t, 0)), s_n(h' + (t, 0))) - \kappa_{D_n} \times \kappa_{D_n}\|_1 < \frac{1}{n}.$$

From now on we will assume that r_n is large so that this condition is satisfied and for each n fix $s_n: H_n \rightarrow D_n$ satisfying (2.6).

Now we define a map $c_{n+1}: H_n \rightarrow G$ by setting $c_{n+1}(h) := s_n(h)\phi_n(h)$. We put $C_{n+1} := c_{n+1}(H_n)$.

The reader should have the following picture in mind. The set F_{n+1} is exactly tiled with the sets $\tilde{F}_n\phi_n(h)$, $h \in H_n$, which may be thought of as ‘windows’. Each F_n has a ‘natural’ translate $F_n\phi_n(h)$ in $\tilde{F}_n\phi_n(h)$ but the translate we actually choose is the natural translate perturbed by a further translation $s_n(h)$ which is chosen in a ‘random’ way and does not move $F_n\phi_n(h)$ out of its window.

It is easy to derive that (1.1)–(1.6) are satisfied for the sequence $(F_n, C_{n+1})_{n=0}^\infty$. Hence the associated (C, F) -action $T = (T_g)_{g \in G}$ of G is well defined on a standard probability space (X, \mathfrak{B}, μ) .

We now state the main result.

Theorem 2.2. *The transformation $T_{(1,0,0)}$ is mixing and 2-fold simple. All non-trivial proper factors of $T_{(1,0,0)}$ are of the form \mathfrak{F}_{G_b} , $0 \neq b \in \mathbb{R}$. All these factors are 2-to-1, prime and pairwise isomorphic.*

We first prove some technical lemmata. After that in Proposition 2.8 we show mixing for $T_{(1,0,0)}$ and in Proposition 2.9 we prove simplicity and describe the centralizer of $T_{(1,0,0)}$. The structure of factors follows then from Veech’s theorem.

Denote by G^0 the subgroup $\mathbb{Z} \times \mathbb{R} \times \{0\}$ of index 2 in G . Given any subset A in G we set $A^0 := A \cap G^0$ and $A^1 := A \setminus A^0$. We will refer to A^0 and A^1 as ‘levels’ of A . We will say that a subset $A \subset G$ is ε -balanced if

$$|\lambda(A^0) - \lambda(A^1)| < \varepsilon \lambda(A).$$

Denote by $\pi_3: G \rightarrow \mathbb{Z}_2$ a natural projection on the third coordinate. Since $\kappa_{D_n} \circ \pi_3^{-1} = \kappa_{\mathbb{Z}_2}$, it follows from (2.6) that

$$(2.7) \quad \|\text{dist}_{h \in H_n} \pi_3 \circ s_n(h) - \kappa_{\mathbb{Z}_2}\|_1 < \frac{1}{n}.$$

¹Here and below by $a \neq b \in A$ we denote two different elements a, b of a set A .

In particular, for any $A^* \subset F_n$ the set $A = A^*C_{n+1}$ is $\frac{1}{n}$ -balanced:

$$(2.8) \quad |\lambda(A^0) - \lambda(A^1)| < \frac{1}{n}\lambda(A).$$

Indeed, since

$$A^0 = \bigsqcup_{h \in s_n^{-1}(G^0)} A^{*0}c_n(h) \sqcup \bigsqcup_{h \in s_n^{-1}(G^1)} A^{*1}c_n(h),$$

$$\lambda(A^0) = \lambda(A^{*0})|s_n^{-1}(G^0)| + \lambda(A^{*1})|s_n^{-1}(G^1)|,$$

and similarly

$$\lambda(A^1) = \lambda(A^{*1})|s_n^{-1}(G^0)| + \lambda(A^{*0})|s_n^{-1}(G^1)|.$$

Hence

$$\begin{aligned} |\lambda(A^0) - \lambda(A^1)| &= |\lambda(A^{*0}) - \lambda(A^{*1})| \left| |s_n^{-1}(G^0)| - |s_n^{-1}(G^1)| \right| \\ &\leq \frac{1}{|H_n|} \lambda(A) \left| |s_n^{-1}(G^0)| - |s_n^{-1}(G^1)| \right|. \end{aligned}$$

It remains to notice that

$$\begin{aligned} \frac{1}{|H_n|} \left| |s_n^{-1}(G^0)| - |s_n^{-1}(G^1)| \right| &\leq \left| \frac{|s_n^{-1}(G^0)|}{|H_n|} - \frac{1}{2} \right| + \left| \frac{|s_n^{-1}(G^1)|}{|H_n|} - \frac{1}{2} \right| \\ &= \|\text{dist}_{h \in H_n} \pi_3 \circ s_n(h) - \kappa_{\mathbb{Z}_2}\|_1 < \frac{1}{n} \end{aligned}$$

by (2.8). It follows that $A = A^*C_{n+1}$ is $\frac{1}{n}$ -balanced for each $A^* \subset F_n$.

Given $h = (h_1, h_2) \in \mathbb{Z}^2$, we let $h^* := (h_1, -h_2)$.

Lemma 2.3. *Let $f = f'\phi_{n-1}(h)$ with $f' \in \tilde{F}_{n-1}$ and $h \in \mathbb{Z}^2$.*

(i) *Suppose $f \in G^\alpha$ and let $\beta := 1 - \alpha$. Let*

$$\begin{aligned} L_n^- &:= \tilde{F}_{n-1}^\alpha \phi_{n-1}(I_{n-2} + h) \sqcup \tilde{F}_{n-1}^\beta \phi_{n-1}(I_{n-2} + h^*) \text{ and} \\ L_n^+ &:= \tilde{F}_{n-1}^\alpha \phi_{n-1}(I_n + h) \sqcup \tilde{F}_{n-1}^\beta \phi_{n-1}(I_n + h^*). \end{aligned}$$

Then $L_n^- \subset fS_n \subset L_n^+$. Hence

$$\frac{\lambda(fS_n \triangle L_n^-)}{\lambda(S_n)} = \bar{o}(1).$$

(ii) *If, in addition, $fS_n \subset F_n$ then for any subset $A = A^*C_{n-1}$ with $A^* \subset F_{n-2}$ we have*

$$\frac{\lambda(AC_n \cap fS_n)}{\lambda(S_n)} = \lambda_{F_{n-1}}(A) + \bar{o}(1).$$

Here $\bar{o}(1)$ means a sequence that goes to 0 as $n \rightarrow \infty$ and does not depend on the choice of A^* in F_{n-2} .

Proof. (i) Suppose $f \in G^0$ (the case $f \in G^1$ is considered in a similar way). We have

$$\begin{aligned} fS_n &= f'\phi_{n-1}(h)\tilde{F}_{n-1}\phi_{n-1}(I_{n-1}) \\ &= f'\tilde{F}_{n-1}^0\phi_{n-1}(h + I_{n-1}) \sqcup f'\tilde{F}_{n-1}^1\phi_{n-1}(h^* + I_{n-1}). \end{aligned}$$

Since $\tilde{F}_{n-1}^0 \tilde{F}_{n-1}^\alpha \subset \bigsqcup_{u \in I_1} \tilde{F}_{n-1}^\alpha \phi_{n-1}(u)$, there exists a partition of \tilde{F}_{n-1}^α into subsets A_u^α , $u \in I_1$, such that $f' A_u^\alpha \subset \tilde{F}_{n-1}^\alpha \phi_{n-1}(u)$ for any u and $\alpha = 0, 1$. Therefore

$$f S_n = \bigsqcup_{u \in I_1} (f' A_u^0 \phi_{n-1}(u)^{-1} \phi_{n-1}(u+h+I_{n-1}) \sqcup f' A_u^1 \phi_{n-1}(u)^{-1} \phi_{n-1}(u+h^*+I_{n-1})).$$

It remains to notice that $\bigsqcup_{u \in I_1} f' A_u^\alpha \phi_{n-1}(u)^{-1} = \tilde{F}_{n-1}^\alpha$.

(ii) Since $f S_n \subset F_n$ and $F_n = \tilde{F}_{n-1} \phi_{n-1}(H_{n-1})$, it follows from (i) that the subsets $K := I_{n-1} + h$ and $K^* := I_{n-1} + h^*$ are contained in H_{n-1} . Therefore

$$\begin{aligned} \frac{\lambda(AC_n \cap f S_n)}{\lambda(S_n)} &= \sum_{k \in H_{n-1}} \frac{\lambda(AC_n(k) \cap f S_n)}{\lambda(S_n)} \\ &= \sum_{k \in H_{n-1}} \frac{\lambda(AC_n(k) \cap L_n^-)}{\lambda(S_n)} + \bar{o}(1) \\ &= \frac{1}{\lambda(S_n)} \sum_{k \in H_{n-1}} \lambda(As_{n-1}(k) \phi_{n-1}(k) \cap \tilde{F}_{n-1}^\alpha \phi_{n-1}(K) \sqcup \tilde{F}_{n-1}^\beta \phi_{n-1}(K^*)) + \bar{o}(1) \\ &= \frac{1}{\lambda(S_n)} \sum_{k \in K} \lambda(As_{n-1}(k) \cap \tilde{F}_{n-1}^\alpha) + \frac{1}{\lambda(S_n)} \sum_{k \in K^*} \lambda(As_{n-1}(k) \cap \tilde{F}_{n-1}^\beta) + \bar{o}(1). \end{aligned}$$

Notice that

$$\lambda(As_{n-1}(k) \cap \tilde{F}_{n-1}^\alpha) = \begin{cases} \lambda(A^\alpha), & \text{if } s_{n-1}(k) \in G^0; \\ \lambda(A^\beta), & \text{if } s_{n-1}(k) \in G^1. \end{cases}$$

In any case, since $A = A' C_{n-1}$ is $\frac{1}{n-2}$ -balanced, we conclude from (2.8) that

$$\lambda(As_{n-1}(k) \cap \tilde{F}_{n-1}^\alpha) = \left(\frac{1}{2} + \bar{o}(1)\right) \lambda(A).$$

In a similar way

$$\lambda(As_{n-1}(k) \cap \tilde{F}_{n-1}^\beta) = \left(\frac{1}{2} + \bar{o}(1)\right) \lambda(A).$$

Hence

$$\begin{aligned} \frac{\lambda(AC_n \cap f S_n)}{\lambda(S_n)} &= \frac{\lambda(A)|K|(1 + \bar{o}(1))}{\lambda(S_n)} + \bar{o}(1) \\ &= \frac{\lambda(A)}{\lambda(F_{n-1})} \cdot \frac{\lambda(F_{n-1})|K|}{\lambda(S_n)} \cdot (1 + \bar{o}(1)) + \bar{o}(1) \\ &= \lambda_{F_{n-1}}(A) \cdot \frac{\lambda(F_{n-1})(2n-1)^2}{(2n+1)^2 \lambda(\tilde{F}_{n-1})} \cdot (1 + \bar{o}(1)) + \bar{o}(1) = \lambda_{F_{n-1}}(A) + \bar{o}(1). \end{aligned}$$

□

Remark 2.4. We note that there is a gap in [DdJ, Lemma 2.3(ii)]. It was stated there that the claim (ii) is true for each subset $A \subset F_{n-1}$. This is not true. However — as was shown in Lemma 2.3(ii) above — the claim is true if $A = A^* C_{n-1}$ for an arbitrary subset $A^* \subset F_{n-2}$. This corrected version of the claim suffices to apply it in the proof of [DdJ, Theorem 2.5] which is the only place in that paper where [DdJ, Lemma 2.3(ii)] was used.

We will also use the following simple lemma.

Lemma 2.5. *Let A , B and S be subsets of finite Haar measure in G . Then*

$$\int_{S \times S} \lambda(Ax \cap By) d\lambda(x) d\lambda(y) = \int_{A \times B} \lambda(aS \cap bS) d\lambda(a) d\lambda(b).$$

Proof. Notice that G is unimodular. Consider two subsets in G^3 :

$$\begin{aligned} \Omega_1 &:= \{(a, x, y) \mid x \in S, y \in S, a \in A \cap Byx^{-1}\} \\ &= \{(a, x, y) \mid a \in A, y \in S, x \in a^{-1}By \cap S\} \text{ and} \\ \Omega_2 &:= \{(a, b, y) \mid a \in A, b \in B, y \in b^{-1}aS \cap S\} \\ &= \{(a, b, y) \mid a \in A, y \in S, b \in B \cap aSy^{-1}\}. \end{aligned}$$

It is clear that the mapping $\Omega_1 \ni (a, x, y) \mapsto (a, axy^{-1}, y) \in \Omega_2$ is 1-to-1 and λ^3 -preserving. Applying Fubini theorem we obtain that

$$\int_{S \times S} \lambda(Ax \cap By) d\lambda(x) d\lambda(y) = \lambda^3(\Omega_1) = \lambda^3(\Omega_2) = \int_{A \times B} \lambda(aS \cap bS) d\lambda(a) d\lambda(b).$$

□

The following lemma is the first step to prove mixing for $T_{(1,0,0)}$. Let $h_0 := (1, 0) \in \mathbb{Z}^2$. Then $\phi_n(h_0) = (1, 0, 0)^{2\tilde{a}_n}$.

Lemma 2.6. *Given a sequence of subsets $H_n^* \subset H_n$ such that $\frac{|H_n^*|}{|H_n|} \rightarrow \delta$ for some $\delta \geq 0$, we let $C_n^* := c_n(H_{n-1}^*)$. Then*

$$(2.9) \quad \sup_{A^*, B^* \in \sigma(\xi_{n-1})} |\mu(T_{\phi_n(h_0)}[A^*C_n^*] \cap [B^*]_{n-1}) - \mu([A^*C_n^*]_n) \mu([B^*]_{n-1})| \rightarrow 0.$$

Proof. Let $A, B \subset F_n$ be ξ_n -measurable. We set $F_n^\circ := \{f \in F_n \mid fS_nS_n^{-1} \subset F_n\}$, $A^\circ := A \cap F_n^\circ$, $B^\circ := B \cap F_n^\circ$, $H'_n := H_n \cap (H_n - h_0)$. It is clear that $\mu(F_n \setminus F_n^\circ) \rightarrow 0$ and $\frac{|H'_n|}{|H_n|} \rightarrow 1$ as $n \rightarrow \infty$. Since $\phi_n(h_0) \in C(G)$ for all $n \in \mathbb{N}$, we have

$$\phi_n(h_0)Ac_{n+1}(h) = As_n(h)\phi_n(h_0 + h) = As_n(h)s_n(h_0 + h)^{-1}c_{n+1}(h_0 + h).$$

whenever $h \in H'_n$. In particular, $\phi_n(h_0)A^\circ c_{n+1}(h) \subset F_{n+1}$ for all $h \in H'_n$. Then

$$\begin{aligned} \mu(T_{\phi_n(h_0)}[A]_n \cap [B]_n) &= \mu(T_{\phi_n(h_0)}[A^\circ]_n \cap [B^\circ]_n) + \bar{o}(1) \\ &= \sum_{h \in H_n} \mu(T_{\phi_n(h_0)}[A^\circ c_{n+1}(h)]_{n+1} \cap [B^\circ]_n) + \bar{o}(1) \\ &= \sum_{h \in H'_n} \mu(T_{\phi_n(h_0)}[A^\circ c_{n+1}(h)]_{n+1} \cap [B^\circ]_n) + \bar{o}(1) \\ &= \sum_{h \in H'_n} \mu([A^\circ s_n(h)s_n(h_0 + h)^{-1}c_{n+1}(h_0 + h)]_{n+1} \cap [B^\circ]_n) + \bar{o}(1) \\ &= \sum_{h \in H'_n} \mu([(A^\circ s_n(h)s_n(h_0 + h)^{-1} \cap B^\circ)c_{n+1}(h_0 + h)]_{n+1}) + \bar{o}(1) \\ &= \frac{1}{|H_n|} \sum_{h \in H'_n} \mu([A^\circ s_n(h)s_n(h_0 + h)^{-1} \cap B^\circ]_n) + \bar{o}(1) \\ &= \frac{1}{|H_n|} \sum_{h \in H'_n} \lambda_{F_n}(A^\circ s_n(h) \cap B^\circ s_n(h_0 + h)) \mu(X_n) + \bar{o}(1) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|H'_n|} \sum_{h \in H'_n} \lambda_{F_n}(A^\circ s_n(h) \cap B^\circ s_n(h_0 + h)) + \bar{o}(1) \\
&= \frac{1}{|H'_n|} \sum_{h \in H'_n} \lambda_{F_n}(A s_n(h) \cap B s_n(h_0 + h)) + \bar{o}(1).
\end{aligned}$$

Let $\nu_n := \text{dist}_{h \in H'_n}(s_n(h), s_n(h + h_0))$. Set $f_{A,B}(x, y) := \lambda_{F_n}(Ax \cap By) = \frac{\lambda(Ax \cap By)}{\lambda(F_n)}$. Notice that

$$\nu_n = \frac{1}{2r_n - 1} \sum_{i=-r_n}^{r_n} \text{dist}_{-r_n \leq t < r_n}(s_n(t, i), s_n(t + 1, i)).$$

It follows from (2.6) that $\|\nu_n - \kappa_{D_n} \times \kappa_{D_n}\|_1 < \frac{1}{n}$. Then by (2.5)

$$\begin{aligned}
\mu(T_{\phi_n(h_0)}[A]_n \cap [B]_n) &= \int_{S_n \times S_n} f_{A,B} d\nu_n + \bar{o}(1) \\
&= \int_{S_n \times S_n} f_{A,B} d\kappa_{D_n} d\kappa_{D_n} + \bar{o}(1) = \frac{1}{\lambda(S_n)^2} \int_{S_n \times S_n} f_{A,B} d\lambda d\lambda + \bar{o}(1),
\end{aligned}$$

Now take $A := A^* C_n^*$ and $B := B^* C_n$ for some ξ_{n-1} -measurable subsets $A^*, B^* \subset F_{n-1}$. We say that elements c and c' of C_n are *partners* if $F_{n-1} c S_n \cap F_{n-1} c' S_n \neq \emptyset$. We then write $c \bowtie c'$. Since $A^* c x \cap B^* c' y = \emptyset$ for $c \not\bowtie c'$, it follows that

$$\begin{aligned}
\int_{S_n \times S_n} f_{A,B} d\lambda d\lambda &= \int_{S_n \times S_n} \lambda_{F_n}(A^* C_n x \cap B^* C_n y) d\lambda(x) d\lambda(y) \\
&= \frac{1}{\lambda(F_n)} \int_{S_n \times S_n} \sum_{C_n^* \ni c \bowtie c' \in C_n} \lambda(A^* c x \cap B^* c' y) d\lambda(x) d\lambda(y).
\end{aligned}$$

Applying Lemma 2.5 we now obtain that

$$\int_{S_n \times S_n} f_{A,B} d\lambda d\lambda = \frac{1}{\lambda(F_n)} \sum_{C_n^* \ni c \bowtie c' \in C_n} \int_{A^* \times B^*} \lambda(ac S_n \cap bc' S_n) d\lambda(a) d\lambda(b).$$

Next, we note that

$$|\lambda(ac S_n \cap bc' S_n) - \lambda(c S_n \cap c' S_n)| \leq 8n \lambda(\tilde{F}_{n-1}) = \bar{o}(1) \lambda(S_n).$$

Each $c \in C_n$ has no more than $2(4n + 1)^2$ partners. Therefore

$$\begin{aligned}
&\mu(T_{\phi_n(h_0)}[A^* C_n^*]_n \cap [B^*]_{n-1}) \\
&= \frac{1}{\lambda(S_n)^2} \sum_{C_n^* \ni c \bowtie c' \in C_n} \int_{A^* \times B^*} \frac{\lambda(c S_n \cap c' S_n) + \lambda(S_n) \bar{o}(1)}{\lambda(F_n)} d\lambda(a) d\lambda(b) + \bar{o}(1) \\
&= \frac{\lambda(A^*) \lambda(B^*)}{\lambda(F_{n-1})^2} \frac{\lambda(F_{n-1})^2}{\lambda(S_n)^2 \lambda(F_n)} \sum_{C_n^* \ni c \bowtie c' \in C_n} (\lambda(c S_n \cap c' S_n) + \lambda(S_n) \bar{o}(1)) + \bar{o}(1) \\
&= \lambda_{F_{n-1}}(A^*) \lambda_{F_{n-1}}(B^*) \theta_n \pm \frac{\lambda(F_{n-1})^2 |H_n^*| 2(4n + 1)^2 \lambda(S_n) \bar{o}(1)}{\lambda(S_n)^2 \lambda(F_n)} + \bar{o}(1) \\
&= \lambda_{F_{n-1}}(A^*) \lambda_{F_{n-1}}(B^*) \theta_n \pm \frac{\lambda(F_{n-1})^2 |H_n^*| 2(4n + 1)^2 \bar{o}(1)}{\lambda(\tilde{F}_{n-1})^2 (2n - 1)^2 |H_n|} + \bar{o}(1) \\
&= \lambda_{F_{n-1}}(A^*) \lambda_{F_{n-1}}(B^*) \theta_n + \bar{o}(1),
\end{aligned}$$

where $\theta_n = \frac{\lambda(F_{n-1})^2}{\lambda(S_n)^2 \lambda(F_n)} \sum_{C_n^* \ni c \bowtie c' \in C_n} \lambda(cS_n \cap c'S_n)$. Substituting $A^* = B^* = F_{n-1}$ and passing to the limit we obtain that $\theta_n \rightarrow \delta$ as $n \rightarrow \infty$. Hence

$$\mu(T_{\phi_n(h_0)}[A^*C_n^*]_n \cap [B^*]_{n-1}) = \mu([A^*C_n^*]_n)\mu([B^*]_{n-1}) + \bar{o}(1).$$

Since $\bar{o}(1)$ does not depend on the choice of A^* and B^* inside F_{n-1} , the claim is proven. \square

Corollary 2.7. *The transformation $T_{(1,0,0)}$ is weakly mixing.*

Proof. Substituting $H_n^* := H_n$ to (2.9) we obtain that

$$\sup_{A^*, B^* \in \sigma(\xi_{n-1})} |\mu(T_{\phi_n(h_0)}[A^*]_{n-1} \cap [B^*]_{n-1}) - \mu([A^*]_{n-1})\mu([B^*]_{n-1})| \rightarrow 0.$$

Since each measurable subset of X can be approximated by $[A^*]_{n-1}$ for large n and ξ_{n-1} -measurable subset $A^* \subset F_{n-1}$, it follows that the sequence $(\phi_n(h_0))_{n=1}^\infty$ is mixing for T , that is $\mu(T_{\phi_n(h_0)}A \cap B) \rightarrow \mu(A)\mu(B)$ for every pair of measurable subsets $A, B \subset X$. \square

Proposition 2.8. *The transformation $T_{(1,0,0)}$ is mixing.*

Proof. We have to show that

$$\lim_{n \rightarrow \infty} \mu(T_{g_n}A \cap B) = \mu(A)\mu(B)$$

for any sequence $(g_n)_{n=1}^\infty$ that goes to infinity in $C(G)$ and every pair of measurable subsets $A, B \subset X$. Let $g_n \in F_{n+1} \setminus F_n$. It suffices to show that a subsequence of $(g_n)_{n=1}^\infty$ is mixing for T . We write $g_n = f_n \phi_n(h_n)$ for some $f_n \in \tilde{F}_n \cap C(G)$ and $h_n \in H_n$. Denote by $z: \mathbb{Z} \rightarrow C(G)$ the natural embedding $z(x) := (x, 0, 0)$. We may assume that $f_n \in z(\mathbb{Z}_+)$ for all n (the case $f_n \in z(\mathbb{Z}_-)$ is considered in a similar way). Let $H'_n := H_n \cap (H_n - h_n)$ and $F'_n := F_n \cap (f_n^{-1}F_n)$. Passing to a subsequence, if necessary, we also may assume without loss of generality that

$$\frac{|H'_n|}{|H_n|} \rightarrow \delta_1 \quad \text{and} \quad \frac{\lambda(F'_n)}{\lambda(F_n)} \rightarrow \delta_2$$

for some $\delta_1, \delta_2 \geq 0$. Partition H_n into three subsets H_n^1, H_n^2 and H_n^3 as follows

$$\begin{aligned} H_n^1 &:= \{h \in H_n \mid g_n F_n c_{n+1}(h) \subset F_{n+1} \phi_{n+1}(h_0)\}, \\ H_n^2 &:= \{h \in H_n \mid g_n F_n c_{n+1}(h) \subset F_{n+1}\}, \\ H_n^3 &:= H_n \setminus (H_n^1 \sqcup H_n^2). \end{aligned}$$

As before $h_0 = (1, 0) \in \mathbb{Z}^2$. Let $C_{n+1}^i := \phi_{n+1}(H_n^i)$. It is clear that $|H_n^3| \leq 4(n+1)(2r_n+1)$ and $|H_n^2 \triangle H'_n| \leq 2r_n+1$. Since $|H_n| = (2r_n+1)^2$, it follows that

$$\frac{|H_n^1|}{|H_n|} \rightarrow 1 - \delta_1, \quad \frac{|H_n^2|}{|H_n|} \rightarrow \delta_1, \quad \frac{|H_n^3|}{|H_n|} \rightarrow 0.$$

Take two ξ_n -measurable subsets $A, B \subset F_n$. Since

$$\mu([AC_{n+1}^3]_{n+1}) = \frac{|C_{n+1}^3|}{|C_{n+1}|} \mu([A]_n) \leq \frac{1}{2r_n+1} \rightarrow 0,$$

we have

$$(2.10) \quad |\mu(T_{g_n}[AC_{n+1}^3]_{n+1} \cap [B]_n) - \mu([AC_{n+1}^3]_{n+1})\mu([B]_n)| \rightarrow 0,$$

so $[F_n C_{n+1}^3]_{n+1}$ is negligible. It suffices to show mixing separately on each of the remaining subsets $[F_n C_{n+1}^1]_{n+1}$ and $[F_n C_{n+1}^2]_{n+1}$.

First, we note that $\phi_{n+1}(h_0)^{-1}g_nF_nC_{n+1}^1 \subset F_{n+1}$. Thus, by (1.12),

$$T_{g_n}[AC_{n+1}^1]_{n+1} = T_{\phi_{n+1}(h_0)}[\phi_{n+1}(h_0)^{-1}g_nAC_{n+1}^1]_{n+1}.$$

By Lemma 2.6 (with $C_{n+1}^* := \phi_{n+1}(h_0)^{-1}\phi_n(h_n)C_{n+1}^1$ and $A^* := f_nA$) we obtain that

$$(2.11) \quad |\mu(T_{g_n}[AC_{n+1}^1]_{n+1} \cap [B]_n) - \mu([AC_{n+1}^1]_{n+1})\mu([B]_n)| \rightarrow 0.$$

It remains to consider the second case involving C_{n+1}^2 . If $\delta_1 = 0$, then obviously

$$(2.12) \quad \mu([AC_{n+1}^2]_{n+1}) \rightarrow 0.$$

Suppose now that $\delta_1 > 0$. Partition A into three subsets A_1 , A_2 and A_3 in the following way: $A_1 := A \cap f_n^{-1}F_n$, $A_2 := A \cap f_n^{-1}F_n\phi_n(h_0)$ and $A_3 := A \setminus (A_1 \sqcup A_2)$. In other words, $f_nA_1 \subset F_n$, $f_nA_2 \subset F_n\phi_n(h_0)$, $f_nA_3 \cap (F_n \sqcup F_n\phi_n(h_0)) = \emptyset$.

Note that

$$(2.13) \quad \mu([A_3C_{n+1}^2]_{n+1}) \leq \mu([A_3]_n) \leq \frac{2n+1}{2r_n+1} \rightarrow 0.$$

For A_1 and A_2 we argue as in the proof of Lemma 2.6. Set $F_n^\circ := \{f \in F_n \mid fS_nS_n^{-1} \subset F_n\}$, $A_1^\circ := A_1 \cap F_n^\circ$, $B^\circ := B \cap F_n^\circ$. We have

$$\begin{aligned} \mu(T_{g_n}[A_1C_{n+1}^2]_{n+1} \cap [B]_n) &= \sum_{h \in H'_n} \mu([\phi_n(h_n)f_nA_1^\circ c_{n+1}(h)]_{n+1} \cap [B^\circ]_n) + \bar{o}(1) \\ &= \sum_{h \in H'_n} \mu([(f_nA_1^\circ s_n(h)s_n(h_n+h))^{-1} \cap B^\circ]c_{n+1}(h)]_{n+1}) + \bar{o}(1) \\ &= \frac{1}{|H_n|} \sum_{h \in H'_n} \mu([f_nA_1^\circ s_n(h)s_n(h_n+h))^{-1} \cap B^\circ]_n) + \bar{o}(1) \\ &= \frac{\delta_1}{|H'_n|} \sum_{h \in H'_n} \lambda_{F_n}(f_nA_1^\circ s_n(h) \cap B^\circ s_n(h_n+h)_n) + \bar{o}(1) \\ &= \delta_1 \int_{S_n \times S_n} f_{A_1 f_n, B} d\nu_n + \bar{o}(1), \end{aligned}$$

where $\nu_n := \text{dist}_{h \in H'_n}(s_n(h), s_n(h_n+h))$ and $f_{A_1 f_n, B}(x, y) = \lambda_{F_n}(A_1 f_n x \cap B y)$.

Write $h_n = (t_n, 0)$. Since $\frac{2r_n - t_n + 1}{2r_n + 1} = \frac{|H'_n|}{|H_n|} \rightarrow \delta_1 > 0$ and

$$\nu_n = \frac{1}{2r_n - 1} \sum_{i=-r_n}^{r_n} \text{dist}_{-r_n \leq t \leq r_n - t_n}(s_n(t, i), s_n(t + t_n, i)),$$

it follows from (2.6) and (2.5) that

$$\mu(T_{g_n}[A_1C_{n+1}^2]_{n+1} \cap [B]_n) = \frac{\delta_1}{\lambda(S_n)^2} \int_{S_n \times S_n} f_{A_1 f_n, B} d\lambda d\lambda + \bar{o}(1).$$

Now take $A := A^*C_n^*$ and $B := B^*C_n$ for some ξ_{n-1} -measurable subsets $A^*, B^* \subset F_{n-1}$. Let $C'_n := C_n \cap F'_n$. It follows that $\frac{|C'_n|}{|C_n|} \rightarrow \delta_2$ and $\mu([A_1]_n \triangle [A^*C_n]_n) = \bar{o}(1)$. Hence $\mu([A_1]_n) = \delta_2\mu([A^*]_{n-1}) + \bar{o}(1)$. Arguing as in the proof of Lemma 2.6 we obtain that

$$\mu(T_{g_n}[A^*C'_nC_{n+1}^2]_{n+1} \cap [B^*]_{n-1}) = \delta_2\mu([A^*]_{n-1})\mu([B^*]_{n-1}) + \bar{o}(1).$$

Therefore

$$(2.14) \quad |\mu(T_{g_n}[A_1C_{n+1}^2]_{n+1} \cap [B]_n) - \mu([A_1C_{n+1}^2]_{n+1})\mu([B]_n)| \rightarrow 0.$$

Since $T_{g_n}[A_2]_n = T_{\phi_n(h_n+h_0)}[\phi_n(h_0)^{-1}f_n A_2]$ with $\phi_n(h_0)^{-1}f_n A_2 \subset F_n$, a similar reasoning yields

$$(2.15) \quad |\mu(T_{g_n}[A_2 C_{n+1}^2]_{n+1} \cap [B]_n) - \mu([A_2 C_{n+1}^2]_{n+1})\mu([B]_n)| \rightarrow 0.$$

Since

$$[A^*]_{n-1} = [A^* C_n C_{n+1}^1]_{n+1} \sqcup \bigsqcup_{i=1}^3 [A_i C_{n+1}^2]_{n+1} \sqcup [A^* C_n C_{n+1}^3]_{n+1},$$

it follows from (2.10)–(2.15) that

$$\lim_{n \rightarrow \infty} \sup_{A^*, B^* \in \sigma(\xi_{n-1})} |\mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) - \mu([A^*]_{n-1})\mu([B^*]_{n-1})| = 0.$$

Since ξ_n -measurable cylinders generate the entire σ -algebra \mathfrak{B} as $n \rightarrow \infty$, it follows that $(g_n)_{n=1}^\infty$ is a mixing sequence for T , as desired. \square

Proposition 2.9. *The transformation $T_{(1,0,0)}$ is 2-fold simple and $C(T_{(1,0,0)}) = \{T_g \mid g \in G\}$.*

Proof. Take an ergodic joining $\nu \in J_2^c(T_{(1,0,0)})$. Let $K_n := [-\frac{a_n}{n^2}, \frac{a_n}{n^2}]_{\mathbb{Z}}$, $J_n := [-\frac{r_n}{n^2}, \frac{r_n}{n^2}]_{\mathbb{Z}}$ and $\Phi_n := K_n + 2\tilde{a}_n J_n$. We claim that ν -a.e. point $(x, y) \in X \times X$ is generic for $T_{(1,0,0)} \times T_{(1,0,0)}$, i.e. for all cylinders $A, B \subset \bigcup_{n=1}^\infty \sigma(\xi_n)$ we have

$$(2.16) \quad \nu(A \times B) = \lim_{n \rightarrow \infty} \frac{1}{|\Phi_n|} \sum_{i \in \Phi_n} \chi_A(T_{(i,0,0)}x) \chi_B(T_{(i,0,0)}y).$$

To see this, we first note that $(\Phi_n)_{n=1}^\infty$ is a Følner sequence in \mathbb{Z} . Since

$$\frac{a_n}{n^2} + \frac{2\tilde{a}_n r_n}{n^2} < \frac{\tilde{a}_n(2r_n + 1)}{n^2} < \frac{2a_{n+1}}{(n+1)^2},$$

it follows that $\Phi_n \subset K_{n+1} + K_{n+1}$ and hence $\bigcup_{m=1}^n \Phi_m \subset K_{n+1} + K_{n+1}$. This implies that $|\Phi_{n+1} + \bigcup_{m \leq n} \Phi_m| \leq 3|\Phi_{n+1}|$ for every $n \in \mathbb{N}$, i.e. Shulman's condition [Li] is satisfied for $(\Phi_n)_{n=1}^\infty$. By [Li], the pointwise ergodic theorem holds along $(\Phi_n)_{n=1}^\infty$ for any ergodic transformation. Since $T \times T$ is ν -ergodic, (2.16) holds for ν -a.a. $(x, y) \in X \times X$ and for every pair of cylinders $A, B \subset X$ from $\bigcup_{n=1}^\infty \sigma(\xi_n)$.

Fix a generic point $(x, y) \in X \times X$. Since $x, y \in X_n$ for all sufficiently large n and we have the following expansion

$$\begin{aligned} x &= (f_n, c_{n+1}(h_n), c_{n+2}(h_{n+1}), \dots), \\ y &= (f'_n, c_{n+1}(h'_n), c_{n+2}(h'_{n+1}), \dots) \end{aligned}$$

with $f_n, f'_n \in F_n$, $h_i, h'_i \in H_i$, $i \geq n$. We let $H_n^- := [-(1 - \frac{1}{n^2})r_n, (1 - \frac{1}{n^2})r_n]_{\mathbb{Z}} \subset H_n$. Since the marginals of ν both equal to μ , we may assume without loss of generality that $h_n, h'_n \in H_n^-$. Indeed,

$$\mu(\{x = (f_n, c_{n+1}(h_n), c_{n+2}(h_{n+1}), \dots) \in X_n \mid h_n \notin H_n^-\}) < \frac{2}{i^2},$$

and hence by Borel-Cantelli lemma for μ -a.e. $x \in X_n$ and all but finitely many i we have $h_i \in H_i^-$. Then we may replace $x = (f_n, c_{n+1}(h_n), c_{n+2}(h_{n+1}), \dots) \in X_n$ with $x = (f_n c_{n+1}(h_n) \cdots c_m(h_{m-1}), c_{m+1}(h_m), \dots) \in X_m$ for some $m > n$ if necessary. Similarly, $h'_n \in H_n^-$.

This implies, in turn, that

$$(2.17) \quad f_{n+1} = f_n c_{n+1}(h_n) \in \tilde{F}_n \phi_n(H_n^-) \subset [-c_n, c_n]_{\mathbb{Z}} \times [-c_n, c_n]_{\mathbb{R}} \times \mathbb{Z}_2,$$

where $c_n = \tilde{a}_n(1 + 2r_n(1 - \frac{1}{n^2}))$, and, similarly, $f'_{n+1} \in [-c_n, c_n]_{\mathbb{Z}} \times [-c_n, c_n]_{\mathbb{R}} \times \mathbb{Z}_2$.

Given $g \in \Phi_n$, there are some uniquely determined $k \in K_n$ and $j \in J_n$ such that $g = k + 2\tilde{a}_n j$, i.e. $(g, 0, 0) = (k, 0, 0)\phi_n(j, 0)$. Moreover, $(j, 0) + h_n \in H_n$ since $h_n \in H_n^-$. It also follows from (2.17) that

$$(2.18) \quad (k, 0, 0)f_n s_n S_n^{\pm 1} \subset F_n.$$

Take $g \in \Phi_n$ and calculate $T_{(g,0,0)}x$.

$$x = (f_n, c_{n+1}(h_n), \dots) = (f_n c_{n+1}(h_n), \dots) = (f_n s_n(h_n)\phi_n(h_n), \dots).$$

$$\begin{aligned} (g, 0, 0)f_n s_n(h_n)\phi_n(h_n) &= (k, 0, 0)\phi_n(j, 0)f_n s_n(h_n)\phi_n(h_n) \\ &= (k, 0, 0)f_n s_n(h_n)\phi_n((j, 0) + h_n) \\ &= (k, 0, 0)f_n s_n(h_n)s_n((j, 0) + h_n)^{-1}c_{n+1}((j, 0) + h_n) \\ &= d c_{n+1}((j, 0) + h_n), \end{aligned}$$

where $d := (k, 0, 0)f_n s_n(h_n)s_n((j, 0) + h_n)^{-1} \in F_n$ by (2.18). This means that $T_{(g,0,0)}x = (d, \dots) \in X_n$. Similarly,

$$(g, 0, 0)f'_n s_n(h'_n)\phi_n(h'_n) = d' c_{n+1}((j, 0) + h'_n)$$

with $d' := (b, 0, 0)f'_n s_n(h'_n)s_n((j, 0) + h'_n)^{-1} \in F_n$.

Now take any ξ_{n-2} -measurable subsets $A^*, B^* \subset F_{n-2}$ and set $A := A^* C_{n-1} C_n$, $B := B^* C_{n-1} C_n$.

$$\begin{aligned} \nu([A^*]_{n-2} \times [B^*]_{n-2}) &= \nu([A]_n \times [B]_n) \\ &= \lim_{n \rightarrow \infty} \frac{|\{g \in \Phi_n \mid T_{(g,0,0)}x \in [A]_n, T_{(g,0,0)}y \in [B]_n\}|}{|\Phi_n|} \\ &= \lim_{n \rightarrow \infty} \frac{|\{g \in \Phi_n \mid d \in A, d' \in B\}|}{|\Phi_n|} \\ &= \lim_{n \rightarrow \infty} \frac{1}{|K_n|} \sum_{k \in K_n} \frac{|\{j \in J_n \mid d \in A, d' \in B\}|}{|J_n|} \\ &= \lim_{n \rightarrow \infty} \frac{1}{|K_n|} \sum_{k \in K_n} \zeta_n(A^{-1}(k, 0, 0)f_n s_n(h_n) \times B^{-1}(k, 0, 0)f'_n s_n(h'_n)), \end{aligned}$$

where $\zeta_n := \text{dist}_{j \in J_n}(s_n((j, 0) + h_n), s_n((j, 0) + h'_n))$. We consider separately two cases.

First case. Suppose first that $h_n \neq h'_n$ for infinitely many, say *bad* n . Since $|J_n| \geq \frac{r_n}{n^2}$ it follows from (2.6) that $\|\zeta_n - \kappa_{D_n} \times \kappa_{D_n}\| < \frac{1}{n}$. Moreover, it follows from (2.18) and (2.5) (we need (2.5) for $A^{-1}(k, 0, 0)f_n s_n(h)$) that

$$\kappa_{D_n}(A^{-1}(k, 0, 0)f_n s_n(h)) = \lambda_{S_n}(A^{-1}(k, 0, 0)f_n s_n(h)) + \bar{o}(1).$$

Hence

$$\begin{aligned} &\frac{1}{|K_n|} \sum_{k \in K_n} \zeta_n(A^{-1}(k, 0, 0)f_n s_n(h_n) \times B^{-1}(k, 0, 0)f'_n s_n(h'_n)) \\ &= \frac{1}{|K_n|} \sum_{k \in K_n} \kappa_{D_n}(A^{-1}(k, 0, 0)f_n s_n(h_n))\kappa_{D_n}(B^{-1}(k, 0, 0)f'_n s_n(h'_n)) + \bar{o}(1) \end{aligned}$$

$$= \frac{1}{|K_n|} \sum_{k \in K_n} \lambda_{S_n}(A^{-1}(k, 0, 0)f_n s_n(h_n)) \lambda_{S_n}(B^{-1}(k, 0, 0)f'_n s_n(h'_n)) + \bar{o}(1)$$

Now we derive from Lemma 2.3(ii) that

$$\begin{aligned} \lambda_{S_n}(A^{-1}(k, 0, 0)f_n s_n(h_n)) &= \frac{\lambda(A^{-1}(k, 0, 0)f_n s_n(h_n) \cap S_n)}{\lambda(S_n)} \\ &= \frac{\lambda(A \cap (k, 0, 0)f_n s_n(h_n) S_n)}{\lambda(S_n)} = \lambda_{F_{n-2}}(A^*) + \bar{o}(1) \end{aligned}$$

and, in a similar way, $\lambda_{S_n}(B^{-1}(b, 0, 0)f'_n s_n(h'_n)) = \lambda_{F_{n-2}}(B^*) + \bar{o}(1)$. Hence

$$\nu([A^*]_{n-2} \times [B^*]_{n-2}) = \lambda_{F_{n-2}}(A^*) \lambda_{F_{n-2}}(B^*) + \bar{o}(1) = \mu([A^*]_{n-2}) \mu([B^*]_{n-2}) + \bar{o}(1)$$

for all bad n and all ξ_{n-2} -measurable subsets $A^*, B^* \subset F_{n-2}$. Since any measurable set can be approximated by $[A^*]_{n-2}$, it follows that in this case $\nu = \mu \times \mu$.

Second case. Now we consider the case where $h_n = h'_n$ for all n greater than some N . Then it is easy to see that $y = T_k x$, where $k = f'_N f_N^{-1} \in G$ and then it follows immediately that (x, y) is generic for the off-diagonal joining μ_{T_k} :

$$\begin{aligned} \nu([A]_n \times [B]_n) &= \lim_{n \rightarrow \infty} \frac{1}{|\Phi_n|} \sum_{i \in \Phi_n} \chi_{[A]_n}(T_{(i,0,0)} x) \chi_{[B]_n}(T_{(i,0,0)} T_k x) = \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\Phi_n|} \sum_{i \in \Phi_n} \chi_{[A]_n \cap T_k^{-1}[B]_n}(T_{(i,0,0)} x) = \mu([A]_n \cap T_k^{-1}[B]_n) = \mu_{T_k}([A]_n \times [B]_n) \end{aligned}$$

for all $A, B \in \sigma(\xi_n)$, since ν projects onto μ . Since each measurable set can be approximated by cylinder sets, we deduce that in this case $\nu = \mu_{T_k}$ with $k \in G$. \square

Proof of Theorem 2.2. follows now from Veech's theorem, Propositions 2.8, 2.9 and the fact that \mathfrak{F}_{G_a} and \mathfrak{F}_{G_b} are isomorphic if and only if G_a and G_b are conjugate in G [dJR, Corollary 3.3]. It is clear, that $G_b = hG_a h^{-1}$ with $h = (0, \frac{a+b}{2}, 1)$. \square

Notice that with some additional conditions on s_n in Lemma 2.1 (cf. [Da3, Lemma 2.3]) one can show that $T_{(1,0,0)}$ is actually mixing of all orders.

3. CONCLUDING REMARKS

If we replace $G = \mathbb{Z} \times (\mathbb{R} \rtimes \mathbb{Z}_2)$ with $\Gamma := \mathbb{R} \times (\mathbb{R} \rtimes \mathbb{Z}_2)$ and apply the same construction (with obvious minor changes) we obtain a probability preserving Γ -action R such that the flow $(R_{(t,0,0)})_{t \in \mathbb{R}}$ is 2-fold simple mixing and its centralizer coincides with the entire Γ -action. This gives an example of 2-fold simple mixing *flow* with uncountably many prime factors. By [Ry], each 2-fold simple flow is simple. For the definitions of higher order simplicity we refer to [dJR]. Moreover, since $\mathbb{Z} \subset \mathbb{R}$ is a closed cocompact subgroup, the corresponding \mathbb{Z} -subaction is also 2-fold simple and $C(R_{(1,0,0)}) = \{R_g \mid g \in \Gamma\}$ by [dJR, Theorem 6.1]. Thus we get examples of two nonisomorphic 2-fold simple transformations with uncountably many prime factors: $R_{(1,0,0)}$ is embeddable into a flow while $T_{(1,0,0)}$ is not.

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